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UNUSUAL STABILIZED FINITE ELEMENT METHODS AND RESIDUAL FREE BUBBLES

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SUMMARY

An overview of the *unusual* stabilized finite element method and of the standard Galerkin method enriched with *residual free bubble* functions is presented. For the first method a concrete model problem illustrates its application in advective–diffusive–reactive equations and for the second method it is shown how static condensation of residual free bubbles gives rise to mass lumping and selective reduced integration, which are viewed as numerical tricks and can now be *derived* by the standard Galerkin method without tricks. © 1998 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Stabilized methods and other non-standard techniques of discretization were developed to deal with intricate physical problems governed by singularly perturbed equations and/or difficulties in approximating systems of equations, etc. The idea then was to develop methods that would not completely change the structure of a simple finite element code based on piecewise polynomials of equal-order approximation for all variables. Surprisingly the idea is effective and can be extended to a variety of applications (see References 1–10 and references cited therein).

Wishing to further understand stabilized finite element methods and other non-standard Galerkin finite element methods, we have revisited the Galerkin method using richer subspaces other than piecewise polynomials. The idea is to enlarge the space of piecewise polynomials with functions defined elementwise, such that improved accuracy and stability are achieved, which are also goals shared by stabilized methods. Noting first that streamline diffusion can be obtained by this process,[11] a theory was developed to show that *virtual* bubbles can be constructed to reproduce stabilized methods in a variety of applications and based on piecewise polynomials of all orders.[12] The questions then were as follows: (i) Is there a family of stabilized methods associated with the

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Galerkin method enriched with bubbles? Which one is it? (ii) Is there a systematic procedure to construct these virtual bubbles so that improved discretizations are developed regardless of how those may look at the end? (In other words, dropping the requirement to obtain a stabilized method, let us pursue ultimately accurate and stable methods.)

This paper deals with these two questions that lead to parallel approaches (not necessarily the same in general). First we show that if we enrich piecewise linears by a space of bubble functions spanned by a single basis function on each element, then we are led to *unusual stabilized finite element methods* (US-FEMs for short). This is discussed in Section 2. The other approach is to select a space of bubble functions that is spanned by the exact solution minus its piecewise linear contribution on an element. This is the space of residual free bubbles that was suggested in References 13–15 and crystallized in References 16–18. An approach similar to the latter one was developed independently in Reference 19 motivated by physical arguments and the equivalence to the residual free bubbles idea is presented in Reference 20. We present the residual free bubbles idea and some of its applications in Section 3.

2. US-FEM: THE UNUSUAL STABILIZED FINITE ELEMENT METHOD

Let L denote a linear differential operator and consider the problem of finding a scalar function u in a smooth domain Ω with boundary Γ such that

$$Lu = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \Gamma = \partial\Omega$$

for given smooth functions f and g. The standard Galerkin method for this problem consists of finding u_h such that

$$a(u_h, v_h) = (Lu_h, v_h) = (f, v_h) \quad \forall v_h, \tag{1}$$

where $(\cdot, \cdot)$ denotes the $L_2(\Omega)$ scalar product. Here u_h and v_h are members of the space spanned by piecewise linear polynomials plus bubble functions, i.e.

$$u_h = u_1 + u_b, \tag{2}$$

where the bubble functions are spanned by one basis function φ_K on an element K,

$$u_b = c_K \varphi_K,$$

and satisfy

$$u_b = 0 \quad \text{on } \partial K.$$

We first consider what method (1) implies for the reduced space of polynomials. In other words, we wish to understand what the effect of u_b on the u_1 part of the solution is.

This question is addressed using what is termed in the finite element literature as *static condensation*, which consists of first taking $v_h = \varphi_K$ on K in equation (1) (zero elsewhere):

$$\begin{aligned} a(u_1 + u_b, \varphi_K)_K &= (f, \varphi_K)_K, \\ a(u_1, \varphi_K)_K + a(u_b, \varphi_K)_K &= (f, \varphi_K)_K, \\ (Lu_1, \varphi_K)_K + a(c_K\varphi_K, \varphi_K)_K &= (f, \varphi_K)_K, \\ c_K a(\varphi_K, \varphi_K)_K &= -(Lu_1 - f, \varphi_K)_K. \end{aligned}$$

Thus the unknown coefficient of the bubble can be computed from

$$c_K = \frac{(Lu_1 - f, \varphi_K)_K}{a(\varphi_K, \varphi_K)_K}. \tag{3}$$

The second part of static condensation is to use $v_h = v_1$ in (1):

$$a(u_1 + u_b, v_1) = (f, v_1),$$
$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1),$$
$$a(u_1, v_1) + \sum_K (u_b, L_K^* v_1)_K = (f, v_1),$$

where L_K^* is the adjoint operator associated with L with boundary condition

$$u_b = 0 \quad \text{on } \partial K.$$

Substituting $u_b = c_K \varphi_K$ with c_K given by (3) yields

$$a(u_1, v_1) - \sum_K \frac{(Lu_1 - f, \varphi_K)_K}{a(\varphi_K, \varphi_K)_K} (\varphi_K, L_K^* v_1)_K = (f, v_1). \tag{4}$$

This is the method suggested by static condensation of one bubble.

At this point we may ask to what kind of method this single-bubble method is related. If we bypass the definition of the bubble shape functions φ_K, we wish to consider a simplification of (4) in the form

$$a(u_1, v_1) - \sum_K (Lu_1, \tau L_K^* v_1) = (f, v_1) - \sum_K (f, \tau L_K^* v_1)_K,$$

where τ is a stability constant.

These are unusual stabilized finite element methods (US-FEMs, suggested in Reference 12 and developed in References 21 and 22) that are consistent methods and more stable than Galerkin methods. As a matter of fact, if L is the first-order operator of convection, then clearly this method adds stability to the Galerkin method in the same way as the Galerkin least-squares method does. However, for zeroth- and second-order operators, bubbles seem to prompt *subtraction* of a square term from the Galerkin term. In this case we need to check whether stability is improved on a case-by-case basis. We have checked that this form of stabilized method is effective for advective–diffusive–reactive equations and shown that these methods lead to convergence for usual elements. Let us start with this model. We consider the problem: find a scalar-valued function $u(\mathbf{x})$ defined in $\Omega \subset \mathbb{R}^2$ such that

$$\sigma u + \mathbf{a} \cdot \nabla u - \kappa \Delta u = f \quad \text{in } \Omega, \tag{5}$$
$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{6}$$

where $\mathbf{a}$ is a given solenoidal velocity field (i.e. $\nabla \cdot \mathbf{a} = 0$), σ and κ are given positive constants and $f(\mathbf{x})$ is a given source function.

Here we have

$$Lu = \sigma u + \mathbf{a} \cdot \nabla u - \kappa \Delta u$$

and by substituting this particular L operator in (1) we have the standard Galerkin method: find u_h such that

$$(\sigma u_h, v_h) + (\mathbf{a} \cdot \nabla u_h, v_h) + (\kappa \nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h.$$

We decompose the solution as before to give

$$u_h = u_1 + u_b$$

and using static condensation we get (see equation (3))

$$c_K = -\frac{(\sigma u_1 + \mathbf{a} \cdot \nabla u_1 - f, \varphi_K)_K}{\sigma \|\varphi\|_{0,K}^2 + \kappa \|\nabla \varphi\|_{0,K}^2}.$$

In the second part of static condensation we are led to the method given by

$$a(u_1, v_1) + \sum_K c_K(\varphi_K, L_K^* v_1)_K = (f, v_1)$$

and therefore by substituting the expression for c_K we have

$$a(u_1, v_1) - \sum_K \frac{(\sigma u_1 + \mathbf{a} \cdot \nabla u_1 - f, \varphi_K)_K}{\sigma \|\varphi\|_{0,K}^2 + \kappa \|\nabla \varphi\|_{0,K}^2} (\varphi_K, \sigma v_1 - \mathbf{a} \cdot \nabla v_1)_K = (f, v_1). \tag{7}$$

We wish to consider the following US-FEM: find u_1 such that

$$a(u_1, v_1) - \sum_K (Lu_1, \tau L^* v_1)_K = (f, v_1) - \sum_K (f, \tau L^* v_1)_K$$

or substituting: find u_1 such that

$$B(u_1, v_1) = F(v_1) \quad \forall v_1, \tag{8}$$

where

$$B(u, v) = (\sigma u, v) + (\mathbf{a} \cdot \nabla u, v) + (\kappa \nabla u, \nabla v) - \sum_{K \in \mathscr{C}_h} (\sigma u + \mathbf{a} \cdot \nabla u - \kappa \Delta u, \tau_K(\sigma v - \mathbf{a} \cdot \nabla v - \kappa \Delta v))_K, \tag{9}$$

$$F(v) = (f, v) - \sum_{K \in \mathscr{C}_h} (f, \tau_K(\sigma v - \mathbf{a} \cdot \nabla v - \kappa \Delta v))_K. \tag{10}$$

The stability constant τ is given by the formulae

$$\tau_K = \begin{cases} h_K / 2|\mathbf{a}|_p, & Pe_K \geq 1, \\ h_K^2 / (\sigma h_K^2 + \beta_K), & Pe_K < 1, \end{cases} \qquad Pe_K = \frac{2|\mathbf{a}|_p h_K}{\sigma h_K^2 + \kappa},$$

$$\beta_K = \begin{cases} |\mathbf{a}|_p h_K, & |\mathbf{a}|_p h_K \geq \kappa, \\ \kappa, & |\mathbf{a}|_p h_K < \kappa, \end{cases} \qquad |\mathbf{a}|_p = \begin{cases} \left(\sum_{i=1}^N |a_i(\mathbf{x})|^p \right)^{1/p}, & 1 \leqslant p < \infty, \\ \max_{i=1,N} |a_i(\mathbf{x})|, & p = \infty. \end{cases}$$

Remarks

1. In the case in which $\sigma = 0$ we have an advective–diffusive equation and the method above reduces to SUPG introduced by Hughes and Brooks.[1]
2. In the case in which $\mathbf{a} = \mathbf{0}$ the method above reduces to the method studied by Franca and Farhat in Reference 21, where a convergence analysis is presented (for piecewise linears).
3. This method deals with the more general possibility of having differential operators of orders two, one and zero present in the same equation. This is of interest in applying these methods to the transport equations of turbulent quantities (such as the ones found in the k–ε model), to applications involving chemical reactions, etc.
4. We would like to reiterate that a single bubble in each element prompts *unusual* stabilized methods in that instead of adding a least-squares form of the Euler–Lagrange equations to the Galerkin method, we subtract a term of the type

$$\sum_{K \in \mathscr{C}_h} \tau(Lu - f, L_K^* v)_K,$$

where L is the differential operator associated with the scalar PDE and L_K^* is its adjoint with zero Dirichlet boundary condition in each element. These unusual methods keep the desired additional stability characteristics of Galerkin least-squares methods and do have a non-trivial counterpart within the framework of the Galerkin method using 'virtual' bubbles.

3. RESIDUAL FREE BUBBLES

This seems to be a very promising approach in that a systematic derivation of methods is now possible. To define residual free bubbles, let us consider the standard Galerkin method for

$$Lu = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega.$$

Then we wish to find u_h such that

$$a(u_h, v_h) = (Lu_h, v_h) = (f, v_h) \quad \forall v_h. \tag{11}$$

Here u_h and v_h are piecewise polynomials plus bubble functions, i.e.

$$u_h = u_1 + u_b, \tag{12}$$

where the bubble functions satisfy the differential equations strongly, i.e.

$$Lu_b = -(Lu_1 - f) \quad \text{in } K, \tag{13}$$

subject to zero Dirichlet boundary condition on the element boundary, i.e.

$$u_b = 0 \quad \text{on } \partial K. \tag{14}$$

Problems given by equations (13) and (14) are addressed by solving instead

$$L\varphi_{i,K} = -L\psi_{i,K} \quad \text{in } K, \tag{15}$$

$$\varphi_{i,K} = 0 \quad \text{on } \partial K, \tag{16}$$

where the $\psi_{i,K}$ are the local basis functions for u_1 and

$$L\varphi_{f,K} = f \quad \text{in } K, \tag{17}$$

$$\varphi_{f,K} = 0 \quad \text{on } \partial K. \tag{18}$$

Thus, if $u_{1|K} = \sum_{i=1}^{n_{\text{en}}} c_{i,K}\psi_{i,K}$, then

$$u_{b|K} = \sum_{i=1}^{n_{\text{en}}} c_{i,K}\varphi_{i,K} + \varphi_{f,K}, \tag{19}$$

with the same coefficients $c_{i,K}$.

As before we ask the question: what does method (1) imply for the reduced space of polynomials? (or, what is the effect of u_b on the u_1 part of the solution?).

The answer is to use *static condensation* (as before)—first $v = v_{b,K}$ on K (zero elsewhere):

$$a(u_1 + u_b, v_{b,K})_K = (f, v_{b,K})_K. \tag{20}$$

However, this equation is satisfied automatically owing to our choice of bubbles. Indeed, this equation is the variational equation for

$$Lu_b = -(Lu_1 - f) \quad \text{in } K, \tag{13}$$

using $v = v_{b,K}$ on K (zero elsewhere) as test functions.

Second part of static condensation—use $v = v_1$ in (11):

$$a(u_1 + u_b, v_1) = (f, v_1),$$

$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1).$$

This is the method suggested by static condensation. Using residual free bubbles, the second-term modification due to the bubbles is computed after solving equations (6)–(9).

Let us consider two examples of this approach to show that mass lumping and selective reduced integration are tricks that can be 'explained' by residual free bubbles.

The presentation now follows.[17] The first example is: find a scalar-valued function $u(x)$ defined in $\Omega \subset \mathbb{R}$ such that

$$\sigma u - \kappa u'' = f \quad \text{in } \Omega, \tag{21}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{22}$$

where σ and κ are given positive constants and $f(x)$ is a given source function. Here

$$Lu = \sigma u - \kappa u''$$

and the residual free bubble problems given by (15)–(18) are

$$\sigma\varphi_{i,K} - \kappa\varphi''_{i,K} = -\sigma\psi_{i,K} \quad \text{in} \quad K, \tag{23}$$

$$\varphi_{i,K} = 0 \quad \text{on} \quad \partial K, \tag{24}$$

where the $\psi_{i,K}$ are the basis functions for u_1 and have second derivative zero inside each element and

$$\sigma\varphi_{f,K} - \kappa\varphi''_{f,K} = f \quad \text{in } K, \tag{25}$$

$$\varphi_{f,K} = 0 \quad \text{on } \partial K. \tag{26}$$

The solutions of (23)–(26) with respect to the local co-ordinate $\xi \in [0, h_K]$ are

$$\varphi_{1,K}(\xi) = \frac{\sinh[\sqrt{(\sigma/\kappa)}(h_K - \xi)]}{\sinh[\sqrt{(\sigma/\kappa)}h_K]} - \left(1 - \frac{\xi}{h_K}\right),$$

$$\varphi_{2,K}(\xi) = \frac{\sinh[\sqrt{(\sigma/\kappa)}\xi]}{\sinh[\sqrt{(\sigma/\kappa)}h_K]} - \frac{\xi}{h_K},$$

$$\varphi_{f,K}(\xi) = -\frac{f}{\sigma}[\varphi_{1,K}(\xi) + \varphi_{2,K}(\xi)]$$

for piecewise constant loads.

The Galerkin method given by the second step of static condensation is

$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1) \tag{27}$$

and in this case

$$(\sigma u_1, v_1) + (\kappa u_1', v_1') + \sum_K (\sigma u_{b,K}, v_1)_K = (f, v_1). \tag{28}$$

If we substitute the expression for $u_{b|K} = \sum_{i=1}^{n_{\mathrm{en}}} c_{i,K}\varphi_{i,K} + \varphi_{f,K}$ with the exact results above, then we are led to a system of equations for the unknown constants $c_{i,K}$.

If we write the system of equations for a uniform mesh, then a typical interior node satisfies (after some algebra)

$$A_h\left(\frac{-c_{I-1} + 2c_I - c_{I+1}}{h}\right) + B_h c_I = C_h f,$$

where

$$A_h = \kappa\frac{\sqrt{(\sigma/\kappa)}h_K}{\sinh[\sqrt{(\sigma/\kappa)}h_K]}, \qquad B_h = 2\sqrt{d(\sigma\kappa)}\tanh\left[\frac{1}{2}\sqrt{\left(\frac{\sigma}{\kappa}\right)}h_K\right], \qquad C_h = \frac{1}{\sigma}B_h.$$

This is the method implied by the residual free bubbles approach. The method will give *nodal exact values* for κ, σ, f and h. For small $\sqrt{(\sigma/\kappa)}h_K$ we have

$$A_h \approx \kappa, \qquad B_h \approx \sigma_h, \qquad C_h \approx h,$$

simplifying the method to

$$\kappa\left(\frac{-c_{I-1} + 2c_I - c_{I+1}}{h}\right) + \sigma h c_I = hf, \tag{29}$$

which is identical in form to the equations produced by the standard Galerkin method using piecewise linears with full integration on the second-derivative term and 'mass lumping' in the zeroth-order term.

The presentation now follows Reference 18. The second example is developed to show the appearance of selected reduced integration from residual free bubbles. The Timoshenko beam model is governed by the differential equations (after non-dimensionalization)

$$-\theta'' - \frac{1}{\varepsilon^2}(w' - \theta) = 0 \quad \text{in } \Omega, \qquad -\frac{1}{\varepsilon^2}(w'' - \theta') = f \quad \text{in } \Omega, \tag{30}$$

where a prime denotes differentiation with respect to $x \in \Omega = (0, 1)$, θ and w are the rotation and displacement variables respectively, f is the load and ε is a non-dimensional parameter proportional to the beam thickness.

To (31) we append the clamped boundary conditions (other boundary conditions may be used without major changes in what follows)

$$w(0) = w(1) = 0, \qquad \theta(0) = \theta(1) = 0. \tag{31}$$

The variational formulation corresponding to (30) and (31) is given by: find $\{\theta, w\} \in H_0^1(\Omega)^2$ such that

$$(\theta', \psi') + \frac{1}{\varepsilon^2}(w' - \theta, v' - \psi) = (f, v) \quad \forall\{\psi, v\} \in H_0^1(\Omega)^2, \tag{32}$$

where we use the notation $(f, g) = \int_\Omega fg \, d\Omega$.

Consider a partition of Ω into non-overlapping elements in the usual way. Then the exact solution of our problem can be decomposed into

$$\theta = \theta_1 + \theta_b, \qquad w = w_1 + w_b, \tag{33}$$

where θ_1 and w_1 are spanned by the standard continuous piecewise linears of finite element methods and θ_b and w_b are assumed to satisfy the following differential equations in each element K:

$$\begin{aligned} -\theta_b'' - \frac{1}{\varepsilon^2}(w_b' - \theta_b) &= -\left(-\theta_1'' - \frac{1}{\varepsilon^2}(w_1' - \theta_1)\right), \\ -\frac{1}{\varepsilon^2}(w_b'' - \theta_b') &= -\left(-\frac{1}{\varepsilon^2}(w_1'' - \theta_1') - f\right), \end{aligned} \tag{34}$$

subject to the boundary conditions

$$\theta_b = w_b = 0 \quad \text{on } \partial K. \tag{35}$$

Equations (34) can be rewritten as (note that $\theta_1'' = w_1'' = 0$ in K)

$$-\varepsilon^2\theta_b'' + \theta_b - w_b' = w_1' - \theta_1, \qquad \theta_b' - w_b'' = -\theta_1' + \varepsilon^2 f. \tag{36a, b}$$

From (36a)

$$\theta_b - w_b' = w_1' - \theta_1 + \varepsilon^2 \theta_b''$$

and combining with (36b) we get

$$\theta_b''' = f \quad \text{in } K. \tag{37}$$

Integrating three times (with respect to the local variable in the element, $\xi \in [0, h_K]$, $h_K = x_{i+1} - x_i$, $\xi = x - x_i$) and assuming a piecewise constant load f, and for notation's sake dropping the subscripts for h and f (nowhere do we need to assume that h_K is constant in what follows), we get

$$\theta_b(\xi) = \frac{\xi^3}{6} f + c_1 \frac{\xi^2}{2} + c_2 \xi + c_3. \tag{38}$$

Applying the boundary conditions $\theta_b(0) = \theta_b(h) = 0$ above gives

$$\theta_b(\xi) = \frac{\xi}{6} f(\xi^2 - h^2) + c_1 \frac{\xi}{2}(\xi - h). \tag{39}$$

Using this expression in (36a), after one integration we get

$$\begin{aligned} w_b(\xi) = \int_0^{\xi} \theta_1(t)\,\mathrm{d}t - w_1(\xi) - \varepsilon^2 \left(\frac{f}{6}(3\xi^2 - h^2) + \frac{c_1}{2}(2\xi - h) \right) \\ + \frac{f}{6}\left(\frac{\xi^4}{4} - \frac{\xi^2}{2} h^2 \right) - \frac{c_1}{12} \xi^2 (3h - 2\xi) + c_4. \end{aligned} \tag{40}$$

Applying the boundary conditions $w_b(0) = w_b(h) = 0$ in (40), we get expressions for the remaining constants c_1 and c_4 and the expressions for the residual free bubble functions are then given by

$$\theta_b(\xi) = f\left(\frac{\xi}{6}(\xi^2 - h^2) + \frac{h\xi}{4}(h - \xi) \right) + \frac{1}{\varepsilon^2 + (h^2/12)} \frac{\xi(\xi - h)}{2} \left[\theta_1\left(\frac{h}{2}\right) - \frac{w_1(h) - w_1(0)}{h} \right], \tag{41}$$

$$\begin{aligned} w_b(\xi) = \xi\left(1 - \frac{\xi}{2h}\right)\theta_1(0) + \frac{\xi^2}{2h}\theta_1(h) + \frac{\xi}{h}[w_1(0) - w_1(h)] \\ - \xi\left(\varepsilon^2 - \frac{\xi^2}{6} + \frac{h\xi}{4}\right)\left\{ \frac{1}{\varepsilon^2 + (h^2/12)} \left[\theta_1\left(\frac{h}{2}\right) - \frac{w_1(h) - w_1(0)}{h} \right] - \frac{hf}{2} \right\} \\ + \frac{f\xi^2}{2}\left(-\varepsilon^2 + \frac{\xi^2}{12} - \frac{h^2}{6} \right). \end{aligned} \tag{42}$$

If we take the test functions $\psi = \psi_1$ and $v = v_1$, where ψ_1 and v_1 are spanned by continuous piecewise linears, then using decomposition (33) the variational formulation (32) can be rewritten as

$$(\theta_1', \psi_1') + \frac{1}{\varepsilon^2}(w_1' - \theta_1, v_1' - \psi_1) - (f, v_1) + \frac{1}{\varepsilon^2}(w_b' - \theta_b, v_1' - \psi_1) = 0, \tag{43}$$

where, by integration by parts, we used that

$$(\theta_b', \psi_1') = \sum_K (\theta_b', \psi_1')_K = \sum_K [(\theta_b, \psi_1')_{\partial K} - (\theta_b, \psi_1'')_K] = 0.$$

Note that (43) consists of the Galerkin method for equal-order piecewise linear approximations for θ and w (without tricks, using full integration) plus a 'perturbation term' that we need to compute based on the bubble functions given by (41) and (42). First by (41) and (42) we compute

$$\begin{aligned} w_b' - \theta_b = {} & \theta_1(0) + \frac{\xi}{h}[\theta_1(h) - \theta_1(0)] - \frac{w_1(h) - w_1(0)}{h} \\ & - \frac{\varepsilon^2}{\varepsilon^2 + (h^2/12)}\left[\theta_1\left(\frac{h}{2}\right) - \frac{w_1(h) - w_1(0)}{h}\right] + \varepsilon^2 f\left(\frac{h}{2} - \xi\right). \end{aligned} \tag{44}$$

Note also that

$$w_1' - \theta_1 = \frac{w_1(h) - w_1(0)}{h} - \left(1 - \frac{\xi}{h}\right)\theta_1(0) - \frac{\xi}{h}\theta_1(h). \tag{45}$$

Thus, summing (44) and (45), we get

$$w_1' - \theta_1 + w_b' - \theta_b = \varepsilon^2 f\left(\frac{h}{2} - \xi\right) - \frac{\varepsilon^2}{\varepsilon^2 + (h^2/12)}\left[\theta_1\left(\frac{h}{2}\right) - \frac{w_1(h) - w_1(0)}{h}\right]. \tag{46}$$

Therefore, using (46), the variational formulation given by (43) reduces to

$$\begin{aligned} (\theta_1', \psi_1') + \sum_K \frac{1}{\varepsilon^2 + (h_K^2/12)}\left(\frac{w_1(h_K) - w_1(0)}{h_K} - \theta_1\left(\frac{h_K}{2}\right), v_1' - \psi_1\right)_K \\ = (f, v_1) + \sum_K f_K\left(\xi - \frac{h_K}{2}, v_1' - \psi_1\right)_K, \end{aligned} \tag{47}$$

where we have introduced the subscripts on h and the piecewise constant load f. This can also be rewritten as

$$(\theta_1', \psi_1') + \sum_K \frac{1}{\varepsilon^2 + (h_K^2/12)}(w_1' - R\theta_1, v_1' - \psi_1)_K = (f, v_1) + \sum_K f_K\left(\xi - \frac{h_K}{2}, v_1' - \psi_1\right)_K, \tag{48}$$

where R stands for a reduced integration operator.

Formulation (48) was *derived* using full integration throughout and by construction its solution is nodally exact. The final form is identical to applying the following tricks to the standard variational formulation.

(i) Use one-point reduced integration on the shear energy term.
(ii) Replace its coefficient $1/\varepsilon^2$ by $1/(\varepsilon^2 + (h_K^2/12))$ in each element.
(iii) Correct the right-hand side as in equation (48) for piecewise constant loads.

To emerge with this collection of 'tricks' requires ingenuity and for the first two tricks different arguments have been given before by several authors.

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